Waves and the One-Dimensional Wave Equation

Earlier we talked about the waves on a pond. Before we start looking specifically at sound waves, let's review some general information about waves.

Types

There are two general classifications of waves, longitudinal and transverse:

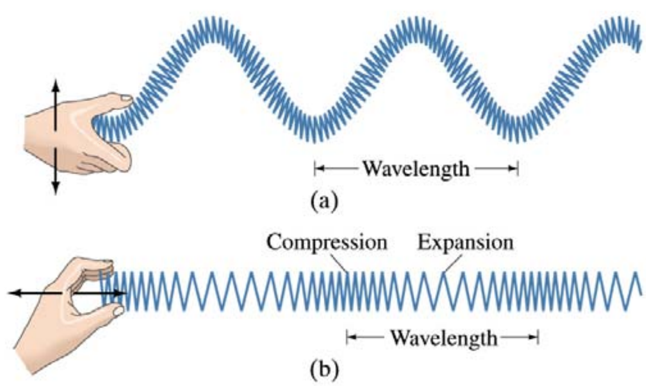
Transverse Wave – A traveling wave in which the particles of the disturbed medium move perpendicularly to the wave velocity. An example is the wave pulse on a stretched rope that occurs when the rope is moved quickly up and down.

Longitudinal Wave – A traveling wave in which the particles of the medium undergo displacement parallel to the direction of the wave motion. Sound waves are longitudinal waves.

One thing to note is that some waves exhibit characteristics of both types of waves. The waves on our pond are a combination of both types.

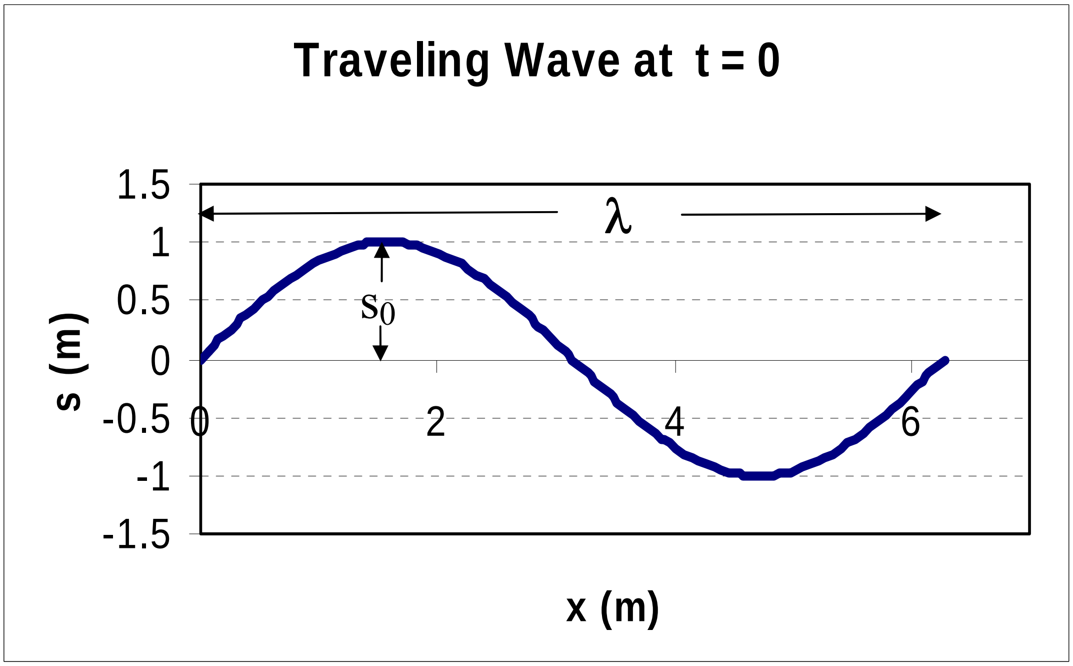
Transverse:

Longitudinal:



Characteristics

Just like the periodic motion of the simple harmonic oscillator, waves have certain characteristics. The ones we will concentrate on are the frequency, period, wave speed and the wavelength. Recall from SP211, a picture of a transverse wave in a medium at some time, maybe t=0 sec.



We wrote an equation to describe this picture:

$$s(x) = s_0 \sin\left(\frac{2\pi}{\lambda}x\right)$$

where:

s = particle displacement – Distance that the fluid particle is moved from its equilibrium position at any time, t.

s_o = maximum particle displacement or amplitude

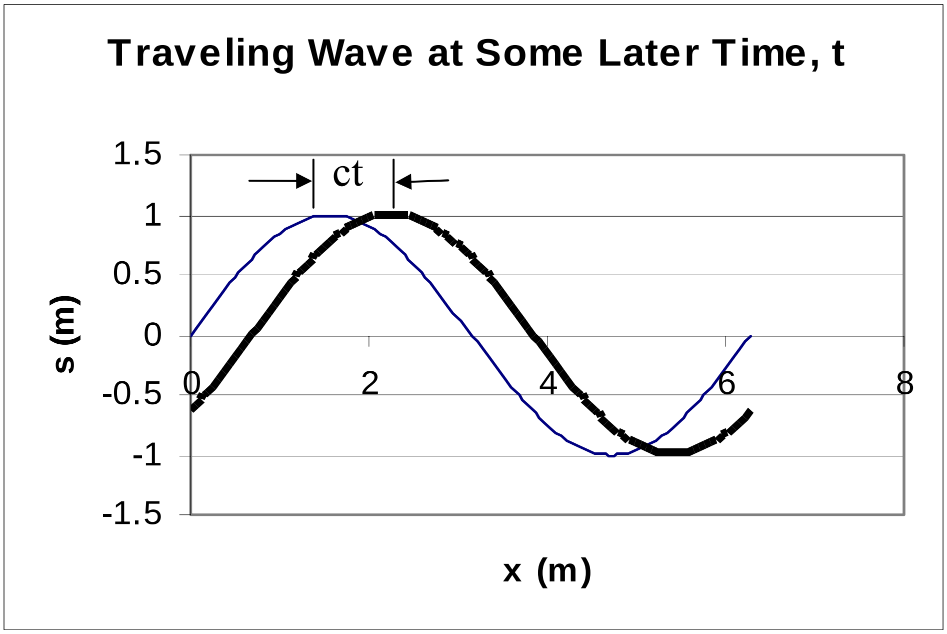
λ = distance over which the wave begins to repeat

k = $\frac{2\pi}{\lambda}$ = a conversion factor that relates the change in phase (angle) to a spatial displacement. We call k the wavenumber.

When we let this wave begin to move to the right with a speed, c, the position is shifted in the governing equation from x to x-ct.

$$s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Below is a picture of the same traveling wave shown at some later time, t.



Now, if instead of taking a snap shot of the wave in the medium at two different times, what if we had set a sensor somewhere in space – maybe at x = 0 m, and recorded the wave's displacement over time. The equation governing the wave would become:

$$s(0,t) = s_o \sin\left[\frac{2\pi}{\lambda}(0-ct)\right] = -s_o \sin\left[\frac{2\pi c}{\lambda}t\right] = -s_o \sin\left[\frac{2\pi}{T}t\right] = -s_o \sin[\omega t]$$

where

T = period – Time to complete one cycle.

$c = \frac{\lambda}{T}$ = wave velocity – Distance that wave energy travels per unit time.

$\omega = \frac{2\pi}{T}$ = a conversion factor that relates the change in phase (angle) to a temporal displacement. We call ω the angular frequency.

$f = \frac{1}{T}$ = frequency, is the inverse of the period. It is the number of cycles per unit time that pass the origin.

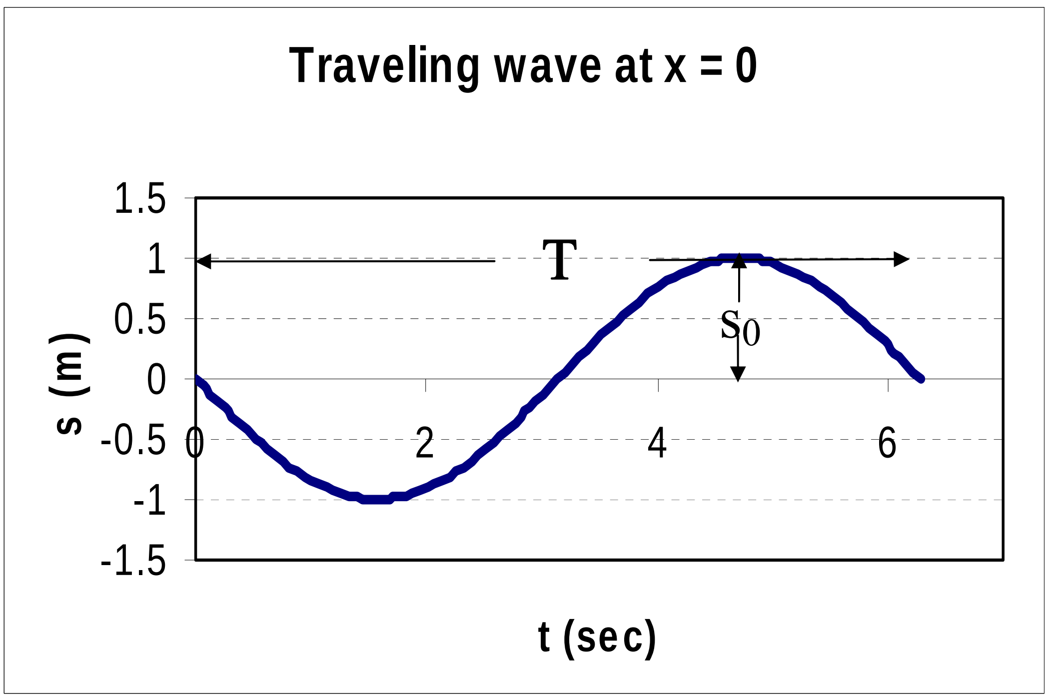
Note that we have employed a similar strategy regarding the group of constants in front of the time variable that we used when discussing the wavenumber, k. Since the wave repeats every 2π change in phase and that corresponds to a time period, T, angular frequency, $\omega=2\pi /T$, is nothing more than a conversion factor from time to phase angle. The symmetry with wavenumber is striking causing many people to identify the wave number as the "special frequency" and to specifically refer to angular frequency, ω, as the "temporal frequency"

To be clear, the speed of the wave c, is not the speed of the medium. It is the speed of the wave disturbance envelope and is often called the "phase speed." It is the speed you would need to run next to the medium in order to stay in phase with a point on the disturbance.

The speed of the medium is also called the particle speed and is found by taking the derivative of the displacement with respect to time.

$u = \frac{\partial s}{\partial t}$ = particle speed – Distance that the medium travels per unit time.

Note that the average value of the particle velocity over any cycle is zero.



Putting these three pictures together, we have an expression for a traveling wave in a medium

$$s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right]$$

or more compactly,

$$s(x,t) = s_o \sin[kx - \omega t]$$

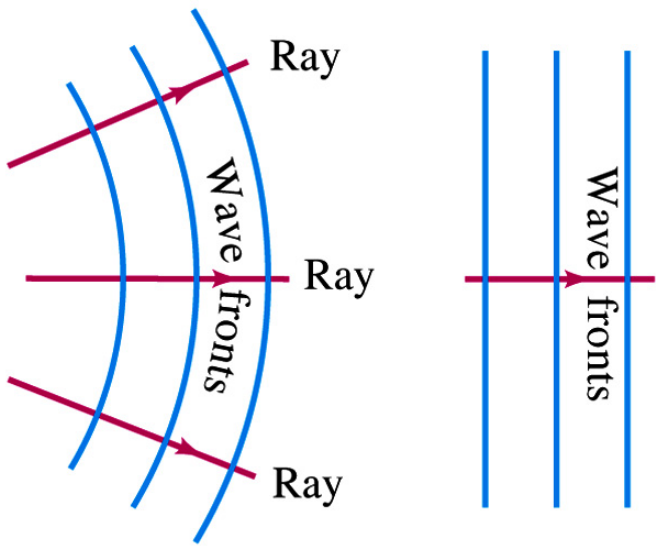
We also have a new way of defining the speed of the wave. It makes good sense that the wave speed is the distance the wave travels in one cycle (the wavelength) divided by the time it takes the wave to complete one cycle. It is a simple matter to substitute the frequency, f, for the period:

$$c = \frac{\lambda}{T} = f\lambda$$

The wave speed can also be calculated from the angular frequency and the wavenumber:

$$c = \frac{\lambda}{T}\left(\frac{2\pi}{2\pi}\right) = \frac{\omega}{k}$$

We call waves modeled using this result “plane waves” because in three dimensions the locus of points all having the same phase are planes. We call these planes “wavefronts” and often draw them as lines on a page separated by one wavelength. In fact, the wavefronts are actually parallel planes. We also find it convenient to show the direction the wave is traveling using a “ray” which is constructed perpendicular to the wavefronts.



Spherical Wave Plane Wave

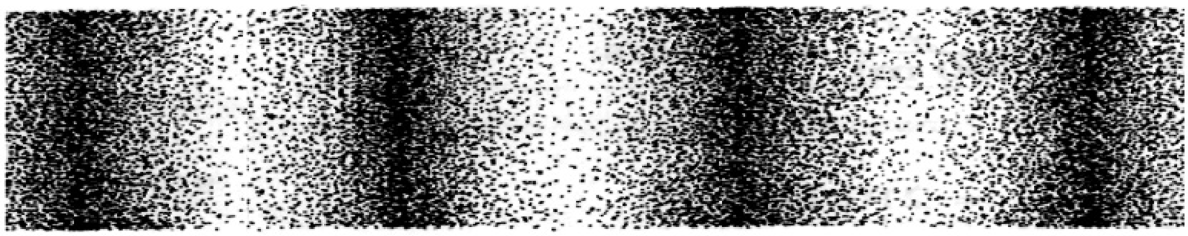
Sound Waves

When sound travels in a fluid, i.e a gas or a liquid, the displacement must be in the longitudinal direction because fluids are poor at transmitting the shear forces necessary to sustain a transverse wave. Below is a cartoon of the longitudinal displacement of a sound wave.

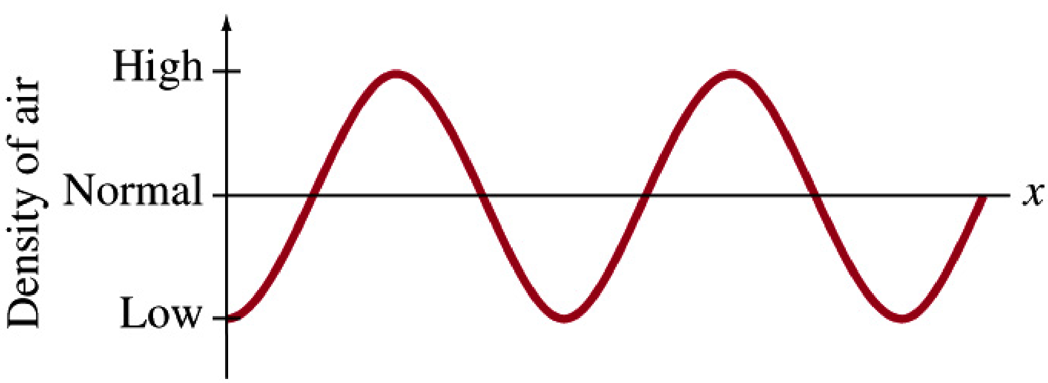
We call the locations where the fluid is displaced into clumps of closely spaced molecules condensations (high density) and the locations where the fluid molecules are sparsely spaced, rarefactions (low density).

The intermolecular forces tend to push out on each other at the condensations just as a compressed spring pushes back on a mass. The gas laws suggest the high density regions of a gaseous fluid are at a higher pressure (force per unit area) and the low density regions are at a lower pressure. The same is true for liquid fluids.

In addition to describing sound waves in fluids by the displacement of the molecules, we can also describe the wave by the velocity of the molecules or the variations in density and pressure.



(a)



(b)

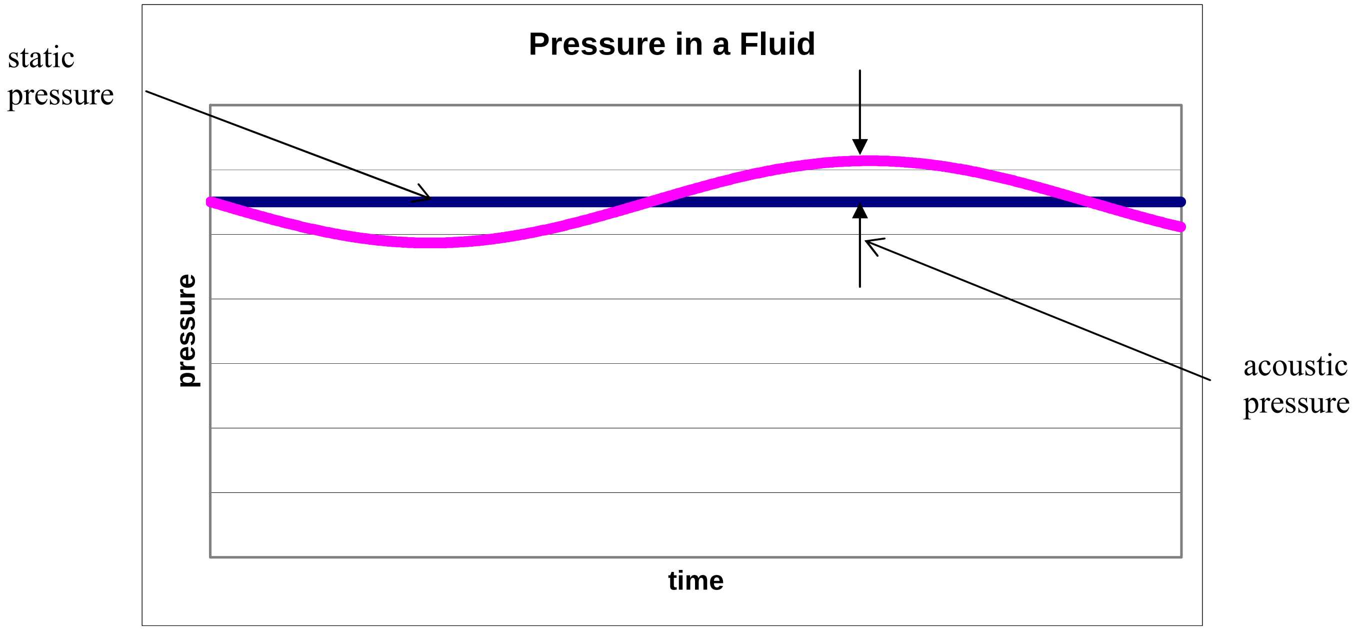
Acoustic Presssure

In the case of pressure, static pressure from the height of the column of fluid above the wave are always present. This force is constant with time. In SP211 we learned how to calculate this pressure, p, using the following equation:

$$p = p_o + \rho g h$$

where ρ is the density of the fluid and h is the height of the fluid column.

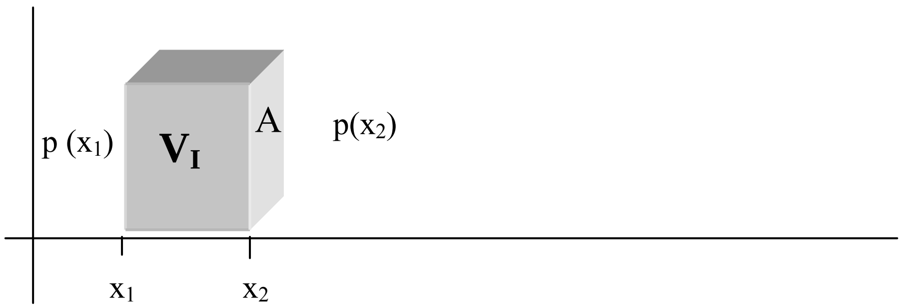
The acoustic pressure due to the condensations and rarefactions sits on this static pressure and oscillates around it due to the presence of the acoustic wave motion. While we could consider the entire pressure variation in describing an acoustic wave, we will, by convention, instead consider only the pressure variation from the static pressure.



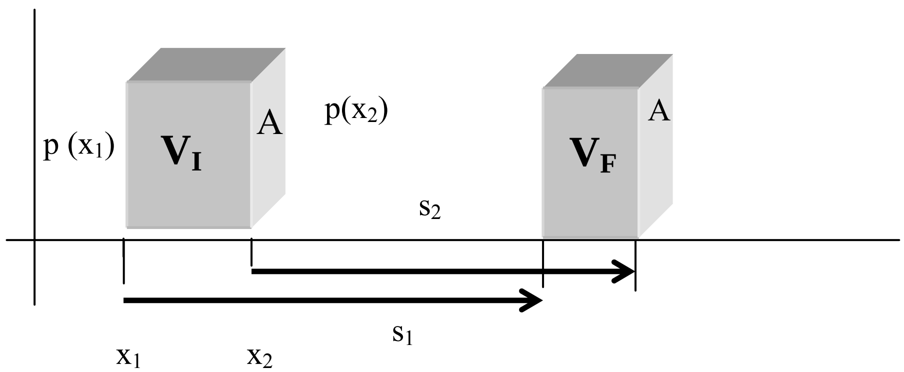
We saw that simple harmonic motion has a governing differential equation called the "equation of motion" whose solution gives the position of a mass as a function of time. In the case of a traveling wave, there is an analogous equation whose solution describes the medium's particle displacement as a function of position and time. This partial differential equation is known as "the wave equation." In the next section we will show how the wave equation follows directly from some fundamental Physics principles.

Sound Waves in a medium – the wave equation

Initial Position – V_I



Later Position – V_F



To derive the one-dimensional wave equation, let's look at the motion of a small volume of fluid. We can relate its motion to the spring-mass system from the previous section. If we apply a pressure gradient to the fluid volume, V_I, (such as an acoustic pressure from an acoustic wave) it will move and compress the volume of fluid. The pressure on the left face of the fluid block is $p_1(x_1)$, while that exerted on the right face is $p_2(x_2)$. If there is a differential pressure, Δp, then the fluid block might move to the right, and, as the block accelerates, it will change to volume, V_F. We will make some assumptions regarding the movement of the block:

1. The process is adiabatic – no heat is lost or gained by the presence of the acoustic wave. This is a reasonable assumption because for acoustic wave frequencies in the ocean, the wavelength is too long and thermal conductivity of seawater too small for significant heat flow to take place.
2. Changes in particle displacement of the fluid from equilibrium are small.
3. The fluid column is not deformed (shear deformation) by differential pressure.

To fully describe the motion of sound in the fluid from first principles, we will examine three well known Physics laws – Newton's Second Law, an equation of state, and conservation of mass. These laws, coupled with the assumptions above provide a robust and powerful model for underwater sound.

Newton's Second Law

Newton's Second Law is customarily used by examining the forces in a particular direction and then summing them as vectors. In the case of our fluid volume, V_I, the forces in the x direction are:

$$\sum F_x = p(x_1)A - p(x_2)A = -\Delta pA$$

This net force across the volume is equal to the mass times the acceleration of the volume. The mass is found by multiplying the initial density by the initial volume ($\Delta x = x_2 - x_1$)

$$m = \rho_I A \Delta x$$

The acceleration in the x direction is the second time derivative of average displacement

$$a_x = \frac{\partial^2 \overline{s}}{\partial t^2}$$

and

$$\overline{s} = \frac{s_1 + s_2}{2}$$

Substituting into Newton's Second Law,

$$\sum F_x = ma_x \quad \text{becomes}$$

$$-\Delta pA = \rho_I A \Delta x \left(\frac{\partial^2 \overline{s}}{\partial t^2} \right) \quad \text{and rearranging gives}$$

$$-\frac{\Delta p}{\Delta x} = \rho_I \left(\frac{\partial^2 \overline{s}}{\partial t^2} \right) \quad \text{or more appropriately}$$

$$\frac{\partial p_a}{\partial x} = -\rho \left(\frac{\partial^2 s}{\partial t^2} \right)$$

In the final result, acoustic pressure was used since the derivative of the static pressure is zero. Additionally, the instantaneous density and displacement for an infinitesimally small volume are substituted.

Equation of State and Conservation of Mass

Even though we think of liquids mostly as incompressible fluids, in reality, they are not. The Bulk Modulus of Elasticity describes how much the volume of the liquid changes for a given change in pressure. In equation form this is:

$$B \equiv -\frac{p(x_2) - p(x_1)}{(V_F - V_I)/V_I} \approx -\frac{p_a}{\Delta V / V_I}$$

The significance of the negative sign in above equation is that when p_a is positive, then $V_F < V_I$ and ΔV is negative.

Using solid geometry we can develop an expression to relate the acoustic pressure to the displacement of the small volume in the above figure. Implied in this argument is the law of conservation of mass. We are not allowing any of the medium to escape the volume, nor are we allowing any additional mass to seep in.

$$V_I = A\Delta x$$

$$V_F = A(\Delta x + \Delta s)$$

(Note: $\Delta s = s_2 - s_1$ is a negative number)

$$\frac{(V_F - V_I)}{V_I} = \frac{\Delta V}{V_I} = \frac{(A(\Delta x + \Delta s) - A\Delta x)}{A\Delta x} = \frac{\Delta s}{\Delta x}$$

Thus substituting in the last two equations and rearranging the definition of the Bulk Modulus of Elasticity:

$$p_a = -B\frac{\Delta V}{V_I}$$

$$p_a = -B\frac{\Delta s}{\Delta x} \quad \text{or more correctly}$$

$$p_a = -B\frac{\partial s}{\partial x}$$

Substituting this last result into our previous relationship between pressure and displacement:

$$\frac{\partial p_a}{\partial x} = -\rho\frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial x}\left(-B\frac{\partial s}{\partial x}\right) = -B\frac{\partial^2 s}{\partial x^2}$$

The One Dimensional Wave Equation

Substituting the conclusion from conservation of mass and equation of state into Newton''s Second Law results in the one-dimensional wave equation that we can use to describe the displacement, s, from their rest position of particles in a medium, with respect to time and position. This equation is a partial differential equation with a solution that varies with time and position. As with the mass-spring system equations, if we can find an equation that satisfies this second order differential equation, the equation could be used to describe the motion of the particles in the medium.

$$\frac{\partial^2 s}{\partial x^2} = \left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$$

One solution that we will use was described above as a plane wave and has the form:

$$s(x,t) = s_o \sin(kx \pm \omega t)$$

Recall that:

s_0 = amplitude of the oscillation or maximum displacement
k = $2\pi/\lambda$ is the wave number
ω = $2\pi f = 2\pi/T$ is the angular frequency
± determines the direction that the wave travels
(+ is for a wave traveling to the left, - is for a wave traveling to the right)

To check the validity of this solution we must take the appropriate second derivatives:

$$\frac{\partial^2}{\partial x^2}\left[s_o \sin(kx - \omega t)\right] = -s_o k^2 \sin(kx - \omega t)$$
$$\frac{\partial^2}{\partial t^2}\left[s_o \sin(kx - \omega t)\right] = -s_o \omega^2 \sin(kx - \omega t)$$

Substitution into the wave equation

$$-s_o k^2 \sin(kx - \omega t) = -\left(\frac{\rho}{B}\right) s_o \omega^2 \sin(kx - \omega t)$$

or

$$k^2 = \left(\frac{\rho}{B}\right)\omega^2$$

Rearranging and recalling that the speed of the wave, $c = \omega/k$,

$$\frac{\omega^2}{k^2} = c^2 = \frac{B}{\rho}$$

This is a fairly profound result. It tells us that the plane wave solution for particle displacement is a "good" solution provided the speed of the wave is not arbitrary, but exactly equal to the square root of the bulk modulus divided by the density. When the bulk modulus and density of water are used, a nominal value for the speed of sound in water is 1500 m/s. This agrees with measured results.

Had we used an equation of state for a gas instead of a liquid, we would have arrived at a similar result following a similar procedure. The plane wave solution would still solve the wave equation, but the wave speed would become:

$$c^2 = \frac{\gamma nRT}{m}$$

When typical room temperature numbers are used, this results in a nominal speed of sound in air of 340 m/s.

The rules of differential equations make no statement about the uniqueness of a solution to the wave equation. Many other solutions exist as well. Had the solution been expressed as a cosine vice a sine, the wave equation would still have been satisfied. Additionally, complex exponentials could have been used as a solution due to Euler's identity.

$$s(x,t) = s_o e^{i(kx-\omega t)}$$

This expression is really shorthand for the real (or imaginary) part of the complex exponential. A Gaussian pulse of the following form also satisfies the wave equation.

$$s(x,t) = s_o e^{-\left(\frac{kx-\omega t}{\omega\tau}\right)^2}$$

Additionally, if a certain frequency wave satisfies the differential equation, all multiples or harmonics of that frequency must also work.

$$s(x,t) = s_o \sin(nkx \pm n\omega t)$$

Rules for differential equations also specify that linear combinations of solutions are also solutions. This is called the principle of superposition. A method using the theory developed by a French mathematician named Fourier will allow disturbances of almost any shape to be constructed using series of harmonic plane waves. These disturbances will still themselves be solutions to the wave equation.

Alternate Views for Describing an Acoustic Wave – The Pressure Field

So far, we have viewed sound moving in a fluid as a harmonic traveling wave, considering only particle displacements. This is not a unique view. Just as an electromagnetic wave can be seen as an oscillating electric field or and oscillating magnetic field, so too can a sound wave be seen as an oscillating pressure field, an oscillating velocity field or an oscillating density field. Of course, the fundamental difference remains that the electromagnetic wave is always a vector field, while the sound wave in a fluid is generally a scalar field.

Using the solution for the wave equation, $s(x,t) = s_0 \sin(kx - \omega t)$, we can find the equations for two of these fields. First the we will find the acoustic pressure. Previously we found the relationship of the acoustic pressure p_a, and the displacement of the small volume from the equation of state. Using this we get:

$$p_a(x,t) = -B\frac{\partial s}{\partial x}$$

$$p_a(x,t) = -B\frac{\partial\left[s_0 \sin(kx - wt)\right]}{\partial x} = -Bs_0 k\cos(kx - wt)$$

The first important observation about the pressure field relative to the displacement field is that they are 90 degrees out of phase with each other. This means that when the particle displacement of the medium is at a maximum, the acoustic pressure is at a minimum. Additionally, when the displacement is zero, the maximum acoustic pressure is:

$$p_{a\max} = Bs_o k = \rho c^2 s_o k$$

By convention, acousticians prefer not to use an engineering modulus, B, instead substituting $B=\rho c^2$.

Alternate Views – The Velocity Field and Specific Acoustic Impedance

The particle velocity is not the wave velocity. The speed that the wave travels is a function of the medium and is a constant. The speed of sound, c, is given by the equations:

$$c = \sqrt{\frac{B}{\rho}} = \frac{\lambda}{T} = f\lambda = \frac{\omega}{k}$$

The particle velocity of the medium, on the other hand tells us how fast the molecules in the fluid are moving. It is found by simply taking the time derivative of the equation describing the position of the medium, the plane wave solution.

$$u(x,t) = \frac{\partial s}{\partial t}$$

$$u(x,t) = \frac{\partial\left[s_0 \sin(kx - \omega t)\right]}{\partial t} = -s_0 \omega \cos(kx - \omega t)$$

where

$$u_{\max} = s_0 \omega = s_0 ck$$

It is noteworthy that for a plane wave, the particle velocity and the particle displacement are 90 degrees out of phase, but that the velocity and acoustic pressure are in phase. We can also find the maximum particle velocity from the characteristics of the wave.

In your electrical engineering classes, you were introduced to a quantity called "impedance." It was the ratio of the driving "force" in a circuit, the voltage, to the rate at which charge passes by a point in the circuit, the current.

$$Z_{\text{electric}} \equiv \frac{\tilde{V}}{\tilde{I}}$$

By analogy, the driving force in an acoustic wave is the pressure and the rate at which particles in the medium pass a particular point is the velocity. It is no accident that we define the specific acoustic impedance as the ratio of the pressure to the particle velocity.

$$z \equiv \frac{p(x,t)}{u(x,t)}$$

For the case of a plane wave we have found expressions for both the pressure and velocity fields.

$$z \equiv \frac{p}{u} = \frac{-\rho c^2 s_0 k \cos(kx - \omega t)}{-s_0 ck \cos(kx - \omega t)} = \rho c$$

The specific acoustic impedance relates the characteristics of a sound wave to the properties of the medium in which it is propagating. Nominal values for the density, ρ, and the wave speed, c, for water are $\rho = 1000\ \text{kg/m}^3$ and $c = 1500$ m/s. Do not be confused into thinking that specific acoustic impedance is always the product of density and the speed of sound. This is only true for a plane wave. For other geometries, for instance a spherically spreading wave, the specific acoustic impedance is a different expression – even in the same fluid.

More on Continuity of Mass – The Density Field

When motivating the wave equation, it was mentioned that the mass in our test fluid volume was not changing. Specifically, the initial mass in position I is the same as that in position F.

$$\rho_I V_I = \rho_F V_F$$

Recalling our expression for the equation of state and substituting,

$$p_a = -B\frac{V_F - V_I}{V_I} = -B\frac{\frac{\rho_I}{\rho_F}V_I - V_I}{V_I} = -B\left(\frac{\rho_I}{\rho_I} - 1\right) = -B\left(\frac{\rho_I - \rho_F}{\rho_F}\right) \approx B\left(\frac{\rho_F - \rho_I}{\rho_I}\right)$$

We find that the fractional change in density, $\left(\frac{\rho_F - \rho_I}{\rho_I}\right)$ is directly proportional to the pressure.

This fractional change in density is called a condensation variable. It is often written,

$$\frac{\rho(x,t) - \rho_o}{\rho_0} = \frac{p_a}{B} = s_o k \cos(kx - \omega t)$$

We have developed four different descriptions for a traveling acoustic plane wave; particle displacement, particle velocity, acoustic pressure and fractional change in density. Particle displacement is 90 degrees out of phase with the other three, but all four discriptions travel with the same wave speed and have the same period and wavelength. All four can be used to properly model acoustic effects.

Energy in a Sound Wave

Missing in the discussion of wave equations and their solution is any mention of energy. We started the semester with a review of simple harmonic (sinusoidal) motion. The reason we did this should be apparent to you by now. As a plane wave traverses any medium, all specific particle locations undergo simple harmonic motion as the wave passes by. Because of this, we can use the basic SP211 equations for kinetic and potential energy of the medium. The only modification is to replace mass with density so as to calculate energy density or energy per unit volume. This is a logical modification since the medium carrying the wave is continuous. It would make no sense to identify a particular piece of mass, nor the total mass. The equations for kinetic and potential energy density in a simple harmonic oscillator are respectively as follows

$$\varepsilon_K = \frac{1}{2}\rho u^2$$

$$\varepsilon_P = \frac{\frac{1}{2}k_{Hooke}s^2}{V} = \frac{\frac{1}{2}m\omega^2 s^2}{V} = \frac{1}{2}\rho\omega^2 s^2$$

Since we have equations for particle displacement and particle velocity, we can simply substitute these into the above.

$$\varepsilon_K = \frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[s_o\omega\cos\left(kx-\omega t\right)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2\cos^2\left(kx-\omega t\right)$$

$$\varepsilon_P = \frac{1}{2}\rho\omega^2 s^2 = \frac{1}{2}\rho\omega^2\left[s_o\sin\left(kx-\omega t\right)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2\sin^2\left(kx-\omega t\right)$$

It should be clear that the total energy is the sum of the potential and kinetic energy and that when the kinetic energy is maximum the potential energy is zero and vice versa. The question of how the energy is partitioned depends on when you ask the question.

The average energy in a simple harmonic oscillator is calculated using the following definition for a periodic function:

$$\left\langle f(t)\right\rangle \equiv \frac{1}{T}\int_0^T f(t)\,dt$$

For kinetic and potential energy we find that since the time average of $\left\langle\sin^2\theta(t)\right\rangle = \left\langle\cos^2\theta(t)\right\rangle = \frac{1}{2}$,

$$\left\langle\varepsilon_K\right\rangle = \frac{1}{2}\rho\omega^2 s_o^2\left\langle\cos^2\left(kx-\omega t\right)\right\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

$$\left\langle\varepsilon_P\right\rangle = \frac{1}{2}\rho\omega^2 s_o^2\left\langle\sin^2\left(kx-\omega t\right)\right\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

This shows that on average, the kinetic energy of a plane wave and the potential energy of a plane wave are the same, each being exactly one half the total energy of the harmonic oscillator. The total average energy density of the wave is then,

$$\langle \varepsilon \rangle = \langle \varepsilon_K \rangle + \langle \varepsilon_P \rangle = \frac{1}{2}\rho u_{max}^2$$

Using the acoustic impedance, $u = \frac{p}{z} = \frac{p}{\rho c}$ allows us to write the total energy in terms of maximum pressure.

$$\langle \varepsilon \rangle = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c^2}$$

Acoustic Intensity

Acoustic intensity, I, is defined as the amount of energy passing through a unit area per unit time as the wave propagates through the medium. As we described in SP211, energy moved per unit time is power which has units of Watts. Intensity then must have units of Watts/m^2.

$$I = \left[\frac{Power}{Area}\right] = \left[\frac{Work}{time} \times \frac{1}{Area}\right] = \left[\frac{Force \times displacement}{Area \times time}\right]$$

$$I = \left[Pressure \times velocity\right]$$

This unit analysis suggests acoustic intensity can be calculated from the product of acoustic pressure and particle velocity.

$$I = p_a u \quad \text{where}$$

$$p_a = p_{a\,max}\sin(kx\text{-}\omega t) \text{ and } u = u_{max}\sin(kx\text{-}\omega t)$$

$$\text{but } u = \frac{p}{\rho c}$$

$$I(x,t) = \frac{p_a^2(x,t)}{\rho c}$$

One important thing to note is that since the <u>acoustic pressure</u> is a time-varying quantity, so is the <u>intensity</u>.

We will use a more meaningful quantity, the time average acoustic intensity. The average intensity of an acoustic wave is the time average of the pressure over a single period of the wave and is given by the equation:

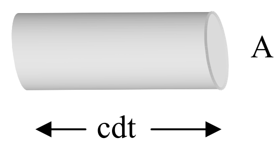
$$\langle I \rangle = \frac{\langle p_a^2 \rangle}{\rho c}$$

Since the time averaged acoustic pressure is $\langle p_a^2 \rangle = \frac{1}{2}p_{a\,max}^2$, the average acoustic intensity can be written:

$$\langle I \rangle = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c}$$

This result looks remarkably similar to the average energy density of a traveling plane wave. In fact this is not accidental. If you consider a plane wave as a cylinder of length cdt and cross section A, the total energy in this cylinder, dE, would be the product of the energy density and the volume.

$$\langle dE \rangle = \langle \varepsilon \rangle Acdt$$



Rearranging we see an alternative expression for average acoustic intensity,

$$\langle I \rangle = \left\langle \frac{1}{A}\frac{dE}{dt} \right\rangle = \langle \varepsilon \rangle c$$

Since $\langle \varepsilon \rangle = \frac{1}{2}\frac{p_{a\max}^2}{\rho c^2}$, the average acoustic intensity is again $\langle I \rangle = \frac{1}{2}\frac{p_{a\max}^2}{\rho c}$.

This result is pleasing in that it agrees with an analogy suggested earlier between voltage and pressure. In your electrical engineering class, you learned that electric power was voltage squared divided by impedance. Average power was found using

$$\langle P \rangle = \frac{1}{2}\frac{V_{max}^2}{Z}$$

Now we have found that average acoustic power per unit area is simply acoustic pressure squared divided by specific acoustic impedance.

$$\langle I \rangle = \frac{\langle P \rangle}{A} = \frac{1}{2}\frac{p_{a\max}^2}{z}$$

This sheds light on why the modifier "specific" precedes acoustic impedance. By analogy, specific acoustic impedance, z, must be acoustic impedance Z, divided by area.

To further make use of electrical engineering backround, time averaged pressure may also be determined by:

$$\langle p_a^2 \rangle = p_{rms}^2$$

$$\sqrt{\langle p_a^2 \rangle} = p_{rms} = \frac{p_{max}}{\sqrt{2}}$$

therefore:

$$\langle I \rangle = \frac{p_{max}^2}{2\rho c} = \frac{p_{rms}^2}{\rho c}$$

Lastly, from this point further, unless otherwise noted, when we refer to the intensity of the wave, we actually mean the time-averaged intensity.

Review and definition:

1. displacement (s) ≡ distance fluid particle moves from equilibrium (meters)
2. period (T) ≡ time required to complete one complete oscillation (seconds)
3. particle velocity (u) ≡ displacement/time (meters/second) = $\frac{\partial s}{\partial t}$
4. wave speed (c) ≡ speed the wave front is moving (meters/second) where $c = \sqrt{\frac{\beta}{\rho}} = \frac{\omega}{k} = \lambda f$
5. frequency (f) ≡ 1/T (Hz or 1/second)
6. angular frequency (ω) ≡ 2πf (radians/second)
7. wave lengths (λ) ≡ distance between same amplitude points of two successive wave fronts (meters)
8. wave number (k) ≡ 2π/λ (1/m)
9. wave fronts ≡ surface over which all particles vibrate in phase
10. acoustic ray ≡ a vector perpendicular to the wave front pointing in the direction of propagation at one specific
11. static pressure (p_s) ≡ pressure of environment minus any changes due to sound wave (Pa or N/m^2)
12. acoustic pressure (p_a) ≡ pressure fluctuations due to presence of wave motion of particle displacement (Pa)
13. instantaneous pressure (p_{tot}) ≡ static plus acoustic pressure at any one instant
14. plane waves ≡ small segment of a spherical wavefront at a long distance from the source
15. rms pressure (p_{rms}) = $\sqrt{\langle p_a^2 \rangle}$ ≡ root mean square value of the acoustic pressure (Pa)
16. Intensity (I) = $p^2_{a\ max}/2\rho c = p^2_{rms}/\rho c$
17. acoustic impedance (z) ≡ p/u =ρc = ρω/k
18. Bulk Modulus of Elasticity (B) ≡ provides relationship between change in pressure to change in volume of unit of fluid
19. density (ρ) ≡ mass contained in a unit volume of fluid (kg/m^3)

Problems

1. A sound wave propagates a point about 50 meters below the surface of a calm sea. The instantaneous pressure at the point is given by: $p = 6x10^5 + 1000\sin(400\pi t)$, where t is in seconds and p in Pascals.
   a) What is the value of static pressure at the point?
   b) What is the value of maximum (or peak) acoustic pressure at the point?
   c) What is the root-mean-square acoustic pressure?
   d) What is the acoustic pressure when t=0, 1.25, 2.5, 3.75, 5.00 milliseconds?
   e) What is the average acoustic intensity of the sound wave? (The density of the water is 1000 kg/m^3 and the sound speed is 1500 m/sec.)
   f) What is the intensity level, L, in dB re 1 μPa?

2. A plane acoustic wave is propagating in a medium of density ρ=1000 kg/m^3. The equation for a particle displacement in the medium due to the wave is given by: $s = (1x10^{-6})\cos(8\pi x - 12000\pi t)$, where distances are in meters and time is in seconds.
   a) What is the rms particle displacement?
   b) What is the wavelength of the sound wave?
   c) What is the frequency?
   d) What is the speed of sound in the medium?
   e) What is the value of maximum (or peak) particle velocity?
   f) What is the value of maximum acoustic pressure?
   g) What is the specific acoustic impedance of the medium?
   h) What is the bulk modulus of the medium?
   i) What is the acoustic intensity of the sound wave?
   j) What is the acoustic power radiated over a 3 m^2 area?

3. A plane acoustic wave is propagating in a medium of density ρ and sound speed c. The equation for pressure amplitude in the medium due to the wave is given by: $p = p_0 \cos(kx - \omega t)$, where p_0 is the maximum pressure amplitude of the sound.
   a) Show that the equation above can be written in the form, $p = p_0 \cos\frac{2\pi}{\lambda}(x - ct)$.
   b) Show that maximum pressure amplitudes (compressions) can be found at the following locations in space: x=nλ+ct where n= 0, 1, 2, 3, …
   c) Show that maximum pressure amplitudes (rarefactions) can be found at the following location in space: x=(n+1/2)λ+ct, where n = 0, 1, 2, 3, …

4. A plane acoustic wave travels to the left with amplitude 100 Pa, wavelength 1.0 m and frequency 1500 Hz; $p_1 = 100Pa\cos\left(\frac{2\pi x}{m} + \frac{3000\pi}{\sec}t\right)$, while another plane wave travels to the right with amplitude 200 Pa, wavelength ½ m and frequency 750 Hz: $p_2 = 200Pa\cos\left(\frac{\pi x}{m} + \frac{1500\pi}{\sec}t\right)$.

a) Find the rms average total pressure. Your answer will not depend on distance x. (Hint: rms average pressure $\equiv \sqrt{\langle P_{total}^2 \rangle}$, where the <> symbol denotes a time average.
b) If $p_1 = p_0 \cos(kx - \omega t)$ and $p_1 = p_0 \cos(kx - \omega t + \phi)$, find the rms average total pressure.

5. Given the following equation for an acoustic wave, originating from a source in the ocean

$$p(x,t) = 8x10^5 \text{ Pa} \sin\left(\frac{2\pi}{13\text{ m}}x - \frac{2\pi[160]}{\text{sec}}t\right)$$

Determine the following:
a) The wavelength
b) The rms pressure of the wave
c) What is the frequency of the wave?
d) The time averaged intensity of the acoustic wave

6. If the particle displacement can be found to be:

$$s(x,t) = 6x10^{-6} \text{ m} \cos\left(\frac{2\pi}{13\text{ m}}x - \frac{2\pi[160]}{\text{sec}}t\right)$$

a) What is the value of the peak particle velocity?
b) What would be the maximum acoustic pressure if the Bulk Modulus of Elasticity of the medium were $2.0x10^9$ N/m^2?

7. If a pressure pulse from a small explosion in water is known to be equal to

$$\mathrm{p} = (1000\mathrm{Pa})\,\mathrm{e}^{-\left(\frac{\mathrm{t}}{0.1\mathrm{sec}}\right)^2} \quad \text{at } x = 0$$

a) Construct a solution to the wave equation for the pulse propagating to the right. This expression must be in the form of a function of x and t.
b) Sketch p(x,t) from part a) for time t = 0, t = 0.1 s, and t = 0.2 s.

8. If an acoustic pressure pulse in water at x = 0 is known to be

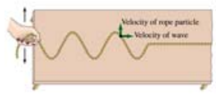
$$\mathrm{p}(\mathrm{t}) = \frac{\mathrm{p_o}}{1+\left(\frac{\mathrm{t}}{\tau}\right)^2} \quad \text{where } \tau = 1 \text{ millesec, } \mathrm{p_o} = 1 \text{ Pa}$$

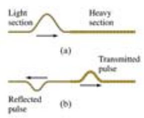
a) Find a wave expression for the pressure pulse traveling in the x-direction to the left.
b) Find an expression for the intensity of the waveform found in part a).

9. What is the speed of sound in yards per second in:
a) air?
b) water?

Waves

- Traveling Waves
  - Types
  - Classification
  - Harmonic Waves
  - Definitions
  - Direction of Travel
- Speed of Waves
- Energy of a Wave





Types of Waves

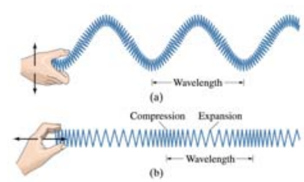
- Mechanical Waves - Those waves resulting from the physical displacement of part of the medium from equilibrium.
- Electromagnetic Waves - Those wave resulting from the exchange of energy between an electric and magnetic field.
- Matter Waves - Those associated with the wave-like properties of elementary particles.

Requirements for Mechanical Waves

- Some sort of disturbance
- A medium that can be disturbed
- Physical connection or mechanism through which adjacent portions of the medium can influence each other.

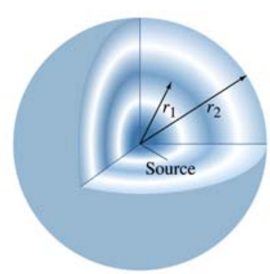
Classification of Waves

- Transverse Waves - The particles of the medium undergo displacements in a direction perpendicular to the wave velocity
  - Polarization - The orientation of the displacement of a transverse wave.
- Longitudinal (Compression) Waves - The particles of the medium undergo displacements in a direction parallel to the direction of wave motion.
  - Condensation/Rarefraction

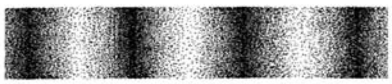


Waves on the surface of a liquid

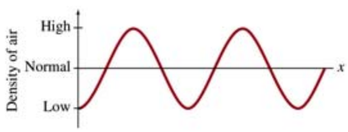
3D Waves



Sound Waves



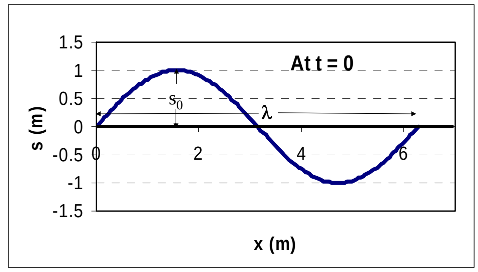
(a)



(b)

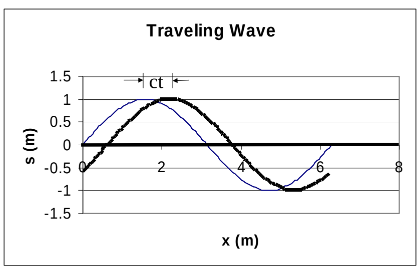
Harmonic Waves

- Transverse displacement looks like:



$$s(x) = s_0 \sin\left(\frac{2\pi}{\lambda}x\right)$$

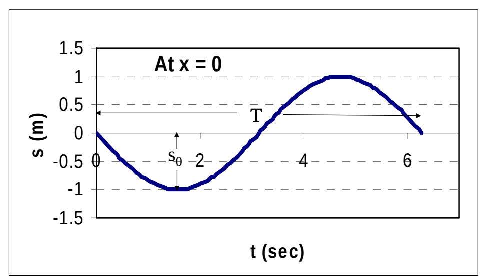
Let the wave move



$$s(x,t) = s_0 \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Standing at the origin

- Transverse displacement looks like:

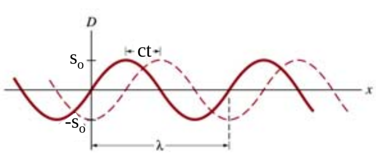


$$s(0,t) = s_o \sin\left[\frac{2\pi}{\lambda}(0 - ct)\right] = s_o \sin\left[-\frac{2\pi c}{\lambda}t\right] = -s_o \sin\left(\frac{2\pi}{T}t\right)$$

Phase Velocity

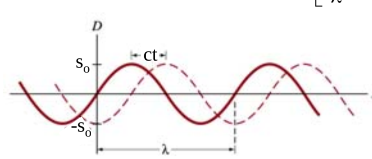
$$c = \frac{\text{distance moved in one cycle}}{\text{time required for one cycle}} = \frac{\lambda}{T} = f\lambda$$

- Wave velocity is a function of the properties of the medium transporting the wave



That negative sign

- Wave moving right $\quad s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right]$
- Wave moving left $\quad s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}x + \frac{2\pi}{T}t\right]$



Alternate notation

$$s(x,t) = s_0 \sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right]$$

$$\boxed{s(x,t) = s_o \sin[kx - \omega t]}$$

Wave number $k = \frac{2\pi}{\lambda}$

Angular frequency $\omega = \frac{2\pi}{T}$

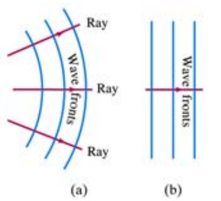
$$c = \frac{\lambda}{T} = \frac{\lambda}{2\pi}\frac{2\pi}{T} = \frac{\omega}{k}$$

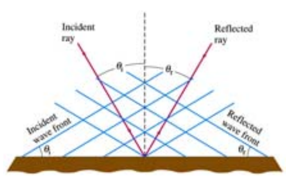
Definitions

- Amplitude - (s_o) Maximum value of the displacement of a particle in a medium (radius of circular motion).
- Wavelength - (λ) The spatial distance between any two points that behave identically, i.e. have the same amplitude, move in the same direction (spatial period)
- Wave Number - (k) Amount the phase changes per unit length of wave travel. (spatial frequency, angular wavenumber)
- Period - (T) Time for a particle/system to complete one cycle.
- Frequency - (f) The number of cycles or oscillations completed in a period of time
- Angular Frequency - (ω) Time rate of change of the phase.
- Phase - ($kx - \omega t$) Time varying argument of the trigonometric function.
- Phase Velocity - (v) The velocity at which the disturbance is moving through the medium

Two dimensional wave motion

Spherical Wave Plane Wave

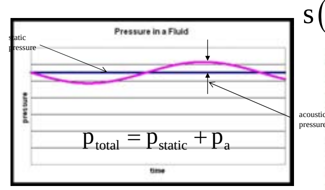


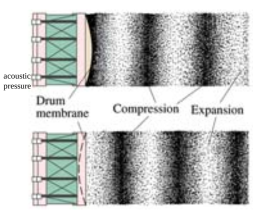


$$\theta_i = \theta_r$$

Acoustic Variables

$$s(x,t) = s_o \sin[kx - \omega t]$$

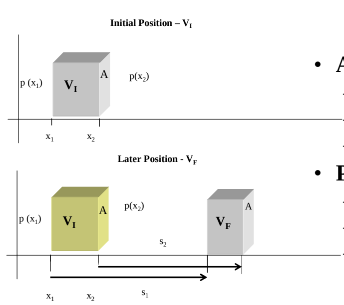




- Displacement
- ParticleVelocity $u = \frac{\partial s}{\partial t}$
- Pressure
- Density $\frac{\rho(x,t) - \rho_o}{\rho_o}$

Condensation = Compression
Rarefaction = Expansion

A microscopic picture of a fluid



- **Assumptions:**
  - Adiabatic
  - Small displacements
  - No shear deformation
- **Physics Laws:**
  - Newton's Second Law
  - Equation of State
  - Conservation of mass

The Wave Equation

Newton's Second Law/ Conservation of Mass $\quad \frac{\partial p_a}{\partial x} = -\rho\left(\frac{\partial^2 s}{\partial t^2}\right)$

Equation of State/ Conservation of Mass $\quad p_a = -B\frac{\partial s}{\partial x}$

PDE – Wave Equation $\quad \frac{\partial^2 s}{\partial x^2} = \left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$

Solutions to differential equations

- Guess a solution
- Plug the guess into the differential equation
  - You will have to take a derivative or two
- Check to see if your solution works.
- Determine if there are any restrictions (required conditions).
- If the guess works, your guess is a solution, but it might not be the only one.
- Look at your constants and evaluate them using initial conditions or boundary conditions.

The Plane Wave Solution

$$s(x,t)=s_o\sin(kx\mp\omega t) \Rightarrow \frac{\partial^2 s}{\partial x^2}=\left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$$

$$-s_o k^2\sin(kx-\omega t)=-\left(\frac{\rho}{B}\right)s_o\omega^2\sin(kx-\omega t)$$

$$k^2=\left(\frac{\rho}{B}\right)\omega^2$$

$$\frac{\omega}{k}=c=\sqrt{\frac{B}{\rho}}$$

General rule for wave speeds

$$c=\sqrt{\frac{\text{Elastic Property}}{\text{Inertial Property}}}$$

Longitudinal wave in a long bar $\quad c=\sqrt{\frac{\text{Young's modulus}}{\text{density}}}=\sqrt{\frac{Y}{\rho}}$

Longitudinal wave in a fluid $\quad c=\sqrt{\frac{\text{Bulk modulus}}{\text{density}}}=\sqrt{\frac{B}{\rho}}$

Sound Speed

$$c=\sqrt{\frac{\text{Bulk modulus}}{\text{density}}}=\sqrt{\frac{B}{\rho}}$$

	Air	Sea Water
Bulk Modulus	1.4(1.01 x 10^{5}) Pa	2.28 x 10^9 Pa
Density	1.21 kg/m^3	1026 kg/m^3
Speed	343 m/s	1500 m/s

Variation with Temperature:

Air $\quad v\approx(331+0.60T)\frac{m}{s}$

Seawater $\quad v\approx(1449.05+4.57T-.0521T^2+.00023T^3)\frac{m}{s}$

Example

- A plane acoustic wave is propagating in a medium of density ρ=1000 kg/m3. The equation for a particle displacement in the medium due to the wave is given by:

  $$s=(1x10^{-6})\cos(8\pi x-12000\pi t)$$

  where distances are in meters and time is in seconds.
- What is the rms particle displacement?
- What is the wavelength of the sound wave?
- What is the frequency?
- What is the speed of sound in the medium?

Alternate Solutions

$$s(x,t)=s_o\cos(kx\pm\omega t)$$

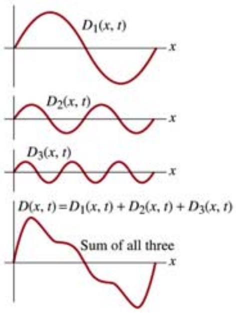
$$s(x,t)=s_o e^{i(kx-\omega t)}$$

$$s(x,t)=s_o e^{-\left(\frac{kx-\omega t}{\omega\tau}\right)^2}$$

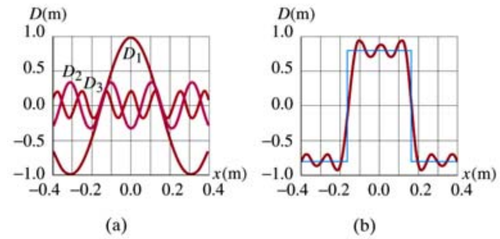
$$s(x,t)=s_o\sin(nkx\pm m\omega t)$$

Superposition

- Waves in the same medium will add displacement when at the same position in the medium at the same time.
- Overlapping waves do not in any way alter the travel of each other (only the medium is effected)



Superposition



- Fourier’s Theorem – any complex wave can be constructed from a sum of pure sinusoidal waves of different amplitudes and frequencies

Alternate Views

Particle Displacement $\boxed{s(x,t) = s_o \sin(kx \pm \omega t)}$

Particle Velocity $u = \dfrac{\partial s}{\partial t} = -s_0 \omega \cos(kx - \omega t)$

Pressure $p_a(x,t) = -B\dfrac{\partial s}{\partial x} = -\rho c^2 s_0 k\cos(kx - wt)$

Density $\dfrac{\rho(x,t) - \rho_o}{\rho_0} = \dfrac{p_a}{B} = -s_o k \cos(kx - \omega t)$

Pitch is frequency

Audible	20 Hz – 20000 Hz
Infrasonic	< 20 Hz
Ultrasonic	>20000 Hz

Middle C on the piano has a frequency of 262 Hz. What is the wavelength (in air)?

1.3 m

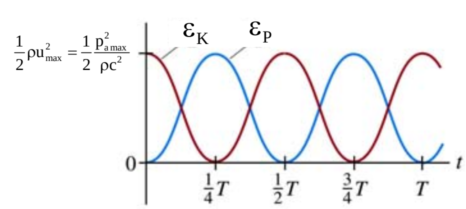
Specific Acoustic Impedance

- Like electrical impedance $Z_{electric} \equiv \dfrac{\tilde{V}}{\tilde{I}}$
- Acoustic analogy
  - Pressure is like voltage
  - Particle velocity is like current
- Specific acoustic Impedance: $z \equiv \dfrac{p(x,t)}{u(x,t)}$
- For a plane wave: $z \equiv \dfrac{p}{u} = \dfrac{-\rho c^2 s_0 k \cos(kx - \omega t)}{-s_0 ck \cos(kx - \omega t)} = \rho c$

Energy Density in a Plane Wave

$$\varepsilon_K = \frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[s_o \omega \cos(kx - \omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2 \cos^2(kx - \omega t)$$

$$\varepsilon_P = \frac{1}{2}\rho\omega^2 s^2 = \frac{1}{2}\rho\omega^2\left[s_o \sin(kx - \omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2 \sin^2(kx - \omega t)$$

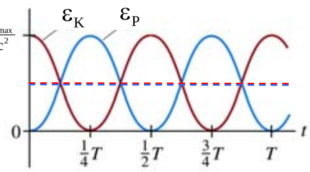


Average Energy Density

$$\langle \varepsilon_K \rangle = \frac{1}{2}\rho\omega^2 s_o^2 \langle \cos^2(kx-\omega t)\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

$$\langle \varepsilon_P \rangle = \frac{1}{2}\rho\omega^2 s_o^2 \langle \sin^2(kx-\omega t)\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

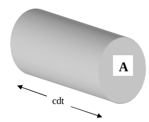
$$\frac{1}{2}\rho u_{max}^2 = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c^2}$$



$$\langle \varepsilon \rangle = \langle \varepsilon_K \rangle + \langle \varepsilon_P \rangle = \frac{1}{2}\rho u_{max}^2$$

Or $\langle \varepsilon \rangle = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c^2}$

Average Power and Intensity



$$\langle dE \rangle = \langle \varepsilon \rangle A c\,dt$$

$$\langle P \rangle = \left\langle \frac{dE}{dt} \right\rangle = \langle \varepsilon \rangle A c$$

$$\langle I \rangle = \frac{\langle P \rangle}{A} = \langle \varepsilon \rangle c = \frac{1}{2}\rho c u_{max}^2 = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c} = \frac{1}{2} p_{a\,max} u_{max}$$

Instantaneous Intensity

$$I(x,t) = p_a(x,t)\,u(x,t) = \frac{[p_a(x,t)]^2}{z} = z[u(x,t)]^2$$

$$I = \left[\frac{Power}{Area}\right] = \left[\frac{Work}{time} \times \frac{1}{Area}\right] = \left[\frac{Force \times displacement}{Area \times time}\right]$$

$$I = [Pressure \times velocity]$$

$$P = \frac{V^2}{Z} = ZI^2 = VI$$

Root Mean Square (rms) Quantities

$$\sqrt{\langle p_a^2 \rangle} = p_{rms} = \frac{p_{max}}{\sqrt{2}}$$

therefore:

$$\langle I \rangle = \frac{p_{max}^2}{2\rho c} = \frac{p_{rms}^2}{\rho c}$$